

# CAYLEY AUTOMATON SEMIGROUPS

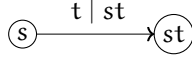
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**ABSTRACT.** In this paper we characterize when a Cayley automaton semigroup is a group, is trivial, is finite, is free, is a left zero semigroup, or is a right zero semigroup.

## 1. INTRODUCTION & MAIN RESULTS

In a bright work [8] Silva & Steinberg introduce the notion of a *Cayley automaton* of a semigroup: having a finite semigroup  $S$ , let  $\mathcal{C}(S)$  be the automaton with state set  $S$  and alphabet set  $S$ , obtained from the Cayley graph of  $S$  by letting the output symbol on the arc leading from  $s$  and labeled by  $t$  to be  $st$ :



Every state from  $\mathcal{C}(S)$  can be viewed as a transformation on the set of all infinite sequences  $S^\infty$ . The semigroup  $\mathbf{C}(S)$  generated by all such transformations, associated to the states of  $\mathcal{C}(S)$ , is obviously the automaton semigroup generated by the automaton  $\mathcal{C}(S)$  (in the sense of [2] and [7]). Silva & Steinberg prove the following

**Theorem 1.1.** *Let  $G$  be a finite non-trivial group. Then  $\mathbf{C}(G)$  is a free semigroup of rank  $|G|$ .*

Under slightly other prospective, Cayley automaton semigroups, derived from monoids, appeared in a work by Mintz [6]. In particular, he proves that if  $S$  is a finite  $\mathcal{H}$ -trivial monoid, then  $\mathbf{C}(S)$  is a finite  $\mathcal{H}$ -trivial semigroup.

The aim of this paper is the following theorems and propositions:

**Theorem 1.2.** *For a finite semigroup  $S$ , the following statements are equivalent:*

- (1)  $\mathbf{C}(S)$  is a group.
- (2)  $\mathbf{C}(S)$  is trivial.
- (3)  $S$  is an inflation of a right zero semigroup by null semigroups.

We prove Theorem 1.2 in Section 3.

**Theorem 1.3.** *Let  $S$  be a finite semigroup. Then  $\mathbf{C}(S)$  is finite if and only if  $S$  is  $\mathcal{H}$ -trivial.*

The proof of sufficiency of Theorem 1.3 in the case when  $S$  is a monoid was proved in [6]. The analogous characterization for the so-called *dual Cayley automaton semigroups* can be found in [5]. The proof of Theorem 1.3 is contained in Section 4.

Making use of Theorem 1.3, in Sections 5 and 6 we will prove the following three propositions:

**Proposition 1.4.** *Let  $S$  be a finite semigroup. Then  $\mathbf{C}(S)$  is a free semigroup if and only if the minimal ideal  $K$  of  $S$  consists of a single  $\mathcal{R}$ -class, in which every  $\mathcal{H}$ -class is not a singleton, and there exists  $k \in K$  such that  $st = skt$  for all  $s, t \in S$ .*

**Proposition 1.5.** *Let  $S$  be a finite semigroup. Then  $\mathbf{C}(S)$  is a right zero semigroup if and only if  $abc = ac$  for all  $a, b, c \in S$ .*

**Proposition 1.6.** *Let  $S$  be a finite semigroup. Then  $\mathbf{C}(S)$  is a left zero semigroup if and only if  $S^2$  is the minimal ideal of  $S$  and if this ideal forms a right zero semigroup.*

In the final Section 7 we will discuss our main results and their corollaries. Before we start proving our statements in the next section we give all necessary notation and lemmas needed for the proofs.

## 2. AUXILIARY LEMMAS

Let  $S = \{s_1, \dots, s_n\}$  be a finite semigroup. In order to avoid confusion, we will denote the states in  $\mathcal{C}(S)$  by an overline:  $s$  is a symbol, and  $\bar{s}$  is a state. The sequences from  $S^\infty$  can be viewed as paths in the infinite  $S$ -rooted tree. So, following [7], we will think about  $\bar{s}$  in terms of the *wreath recursion*:

$$\bar{s} = \lambda_s(\overline{ss_1}, \dots, \overline{ss_n}),$$

where  $\lambda_s : S \rightarrow S$ , defined by  $x \mapsto sx$ , corresponds to the action of  $\bar{s}$  on the first level of the  $S$ -tree. Notice that  $\lambda_s \lambda_t = \lambda_{ts}$  for all  $s, t \in S$ . Hence

$$\bar{s} \cdot \bar{t} = \lambda_{ts}(\overline{ss_1 \cdot ts_1}, \dots, \overline{ss_n \cdot ts_n}).$$

Iterating this formula one obtains that when automaton  $\mathcal{C}(S)$  is in state  $\overline{a_1} \cdots \overline{a_k}$  and reads a symbol  $x$ , it moves to the state

$$q(\overline{a_1} \cdots \overline{a_k}, x) = \overline{a_1 x} \cdot \overline{a_2 a_1 x} \cdots \overline{a_k \cdots a_1 x}.$$

The transformation  $\tau(\overline{a_1} \cdots \overline{a_k})$  on the set  $S$ , correspondent to the action of  $\overline{a_1} \cdots \overline{a_k}$  on the first level of the  $S$ -tree is obviously  $\lambda_{a_k \cdots a_1}$ .

**Lemma 2.1.** *Let  $S$  be a finite semigroup. Then for all  $s, t \in S$ ,  $\bar{s} = \bar{t}$  in  $\mathbf{C}(S)$  if and only if  $\lambda_s = \lambda_t$ .*

*Proof.* Let  $s, t \in S$ . Then  $\bar{s} = \bar{t}$  if and only if  $\lambda_s = \lambda_t$  and  $\overline{sx} = \overline{tx}$  for all  $x \in S$ . Recursing this, we obtain that  $\bar{s} = \bar{t}$  if and only if  $\lambda_s = \lambda_t$  and  $\lambda_{sx} = \lambda_{tx}$  for all  $x \in S$ . It remains to notice that  $\lambda_s = \lambda_t$  implies  $\lambda_{sx} = \lambda_{tx}$  for all  $x \in S$ .  $\square$

Now we calculate Cayley automaton semigroups of special type semigroups:

**Lemma 2.2.** *Let  $L$  be a finite left zero semigroup. Then  $\mathbf{C}(L)$  is a right zero semigroup with  $|L|$  elements.*

*Proof.* Suppose  $\mathcal{C}(L)$  is in state  $\bar{s}$  and reads symbol  $t$ . Then, by the definition of  $\mathcal{C}(L)$ , it outputs  $s$  and moves to the same state  $\bar{s}$ . Thus  $\alpha \cdot \bar{s} = s^\infty$  for all  $\alpha \in L^\infty$ . Hence for any  $s, t \in S$  and  $\alpha \in L^\infty$ :

$$\alpha \cdot (\bar{s} \cdot \bar{t}) = s^\infty \cdot \bar{t} = t^\infty = \alpha \cdot \bar{t},$$

and so  $\bar{s} \cdot \bar{t} = \bar{t}$ . It remains to note that, by Lemma 2.1, if  $s \neq t$ , then  $\bar{s} \neq \bar{t}$ .  $\square$

**Lemma 2.3.** *Let  $S$  be a finite semigroup and let  $R$  be a finite right zero semigroup. Then  $\mathbf{C}(S \times R) \cong \mathbf{C}(S)$ .*

*Proof.* Let  $s \in S$  and  $r, t \in R$ . Then it follows from Lemma 2.1 that  $\overline{(s, r)} = \overline{(s, t)}$  in  $\mathbf{C}(S \times R)$ . Hence  $\mathbf{C}(S \times R)$  coincides with  $T = \langle \overline{(s, r_0)} : s \in S \rangle$  for any fixed  $r_0 \in R$ . It is now easy to check that  $\overline{(s, r_0)} \mapsto \bar{s}$  gives rise to an isomorphism from  $T$  onto  $\mathbf{C}(S)$ .  $\square$

**Corollary 2.4.** *Let  $R$  be a finite right zero semigroup. Then  $\mathbf{C}(R)$  is trivial.*

The proof of the last lemma of this section is easy, it can be found in [5].

**Lemma 2.5.** *Let  $S$  be a finite semigroup and let  $a, b \in S$ . If all  $a, b$  and  $ab$  belong to the same  $\mathcal{D}$ -class of  $S$  then  $ab \in R_a \cap L_b$ .*

### 3. PROOF OF THEOREM 1.2

We recall that a semigroup  $S$  is an inflation of a right zero semigroup  $T$  by null semigroups if  $T \leq S$  and  $S$  can be partitioned into disjoint subsets  $S_t$  (for each  $t \in T$ ) such that  $t \in S_t$  and  $S_u S_t = \{t\}$  for all  $t, u \in S$ .

*Proof of Theorem 1.2.* The proof follows via the chain  $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$ .

$(2) \Rightarrow (1)$  is clear.

$(3) \Rightarrow (2)$ . Let  $S$  be an inflation of a right zero semigroup  $T$ . Then for all  $s, t, x \in S$ , we have  $sx = tx$ . Hence  $\lambda_s = \lambda_t$  and so, by Lemma 2.1,  $\bar{s} = \bar{t}$  for all  $s, t \in S$ . It remains to prove that for any fixed  $s \in S$ , the element  $\bar{s}$  is an idempotent. We have  $\bar{s} = \lambda_s(\bar{s}, \dots, \bar{s})$  and  $\bar{s}^2 = \tau(\bar{s}^2)(\bar{s}^2, \dots, \bar{s}^2)$ . Thus it suffices to prove that  $\tau(\bar{s}^2) = \lambda_s$ . This holds since  $s^2 x = sx$  for all  $x \in S$ .

$(1) \Rightarrow (3)$ . We will prove by induction on  $|S|$  that if  $\mathbf{C}(S)$  is a group, then  $S$  is an inflation of a right zero semigroup by null semigroups. The base case  $|S| = 1$  is trivial. So suppose the implication holds for all semigroups of cardinality  $< |S|$  and that  $\mathbf{C}(S)$  is a group.

Let  $\mathcal{T}_S$  be the transformation semigroup on  $S$ . The subsemigroup  $\langle \lambda_s : s \in S \rangle$  in  $\mathcal{T}_S$  is a homomorphic image of the group  $\mathbf{C}(S)$  and so is a group. This implies that the images and kernels of the mappings  $\lambda_s$  must coincide. These conditions can be translated as

- $sS = tS$  for all  $s, t \in S$  and
- $sx = sy$  if and only if  $tx = ty$ , for all  $s, t, x, y \in S$ .

Notice that the condition that  $sS = tS$  for all  $s, t \in S$  is equivalent to  $sS = S^2$ . The rest of the proof depends on whether  $S^2 = S$  or not.

**Case 1:**  $S^2 = S$ .

Then, by the observations above,  $sS = S^2 = S$  for all  $s \in S$ . So each  $s \in S$ , acting via left-multiplication, permutes  $S$ . Then for any  $s \in S$ , some power of  $s$  is a left identity  $e$  for  $S$ . Then for all  $s, x, y \in S$ , the condition  $sx = sy$  implies  $x = ex = ey = y$ . Hence  $S$  is left cancellative. The condition that  $sS = S$  for all  $s \in S$  implies that  $S$  is right simple.

Therefore  $S$  is a right group and so  $S = G \times R$  for some group  $G$  and a right zero semigroup  $R$ , see [1]. By Lemma 2.3, we have  $\mathbf{C}(G \times R) = \mathbf{C}(G)$ . Since the Cayley automaton semigroup over every non-trivial group is a free semigroup,  $G$  must be trivial. Then  $S \cong R$  is a right zero semigroup and so (3) holds.

**Case 2:**  $S^2 \neq S$ .

The condition of the case means that  $S$  contains indecomposable elements.

Recall that the kernels of all the mappings  $\lambda_s$  coincide. Partition  $S$  into these kernel classes  $A_1, \dots, A_k$  and notice that, for every  $s \in S$ , the equality  $sx = sy$  holds if and only if  $x$  and  $y$  come from the same class. Furthermore, since the mappings  $\lambda_s$  generate a subgroup in  $\mathcal{T}_S$ , it follows that every kernel class  $A_i$  contains an image point, which must of course be an element of  $S^2$  (the image of every  $\lambda_s$  is  $S^2$ ).

The remainder of the proof we will work out in two subcases:

**Subcase a:** for all  $a \in S \setminus S^2$  there exists  $x \in S \setminus \{a\}$  with  $x(S \setminus \{a\}) \neq (S \setminus \{a\})(S \setminus \{a\})$ .

Consider an arbitrary  $a \in S \setminus S^2$  and find the corresponding  $x \in S \setminus \{a\}$ . That  $x(S \setminus \{a\}) \neq (S \setminus \{a\})(S \setminus \{a\})$  means that there exists an element  $uv \notin x(S \setminus \{a\})$  where  $u, v \in S \setminus \{a\}$ . Since  $uS = xS$ , there exists some  $b \in S$  with  $uv = xb$ . Obviously then  $b = a$ . Hence  $xa \notin x(S \setminus \{a\})$ . That is,  $xa \neq xy$  for all  $y \in S \setminus \{a\}$ . This is equivalent to that  $sa \neq sy$  for all  $s \in S$  and  $y \in S \setminus \{a\}$ . The kernel class  $A$  that contains  $a$  is not a singleton, for it must contain an element from  $S^2$  and  $a$  itself is indecomposable. Take an arbitrary  $c \in A \setminus \{a\}$ . Then  $sa = sc$  for any  $s \in S$ , a contradiction.

**Subcase b: There exists  $a \in S \setminus S^2$  such that for all  $x \in S \setminus \{a\}$  there holds  $x(S \setminus \{a\}) = (S \setminus \{a\})(S \setminus \{a\})$ .**

Fix such an  $a$ . Obviously  $T = S \setminus \{a\}$  is a subsemigroup of  $S$ .

We will show now that  $\mathbf{C}(T)$  is a homomorphic image of  $\mathbf{C}(S)$  and thus that  $\mathbf{C}(T)$  is a group. Let  $I$  be the minimal ideal in  $S$ . Then  $I$  is simple and so, being finite, is completely simple. Hence  $I$  is a Rees matrix semigroup. Take now an arbitrary  $i \in I$ . Since  $I$  is a Rees matrix semigroup, there exists  $e \in I$  such that  $ei = i$ . Then  $\tau(\bar{i} \cdot \bar{a}) = \lambda_{ai} = \lambda_{aei} = \tau(\bar{i} \cdot \bar{ae})$  and

$$q(\bar{i} \cdot \bar{a}, x) = \overline{ix} \cdot \overline{aix} = \overline{ix} \cdot \overline{aeix} = q(\bar{i} \cdot \bar{ae}, x)$$

for all  $x \in S$ . Thus  $\bar{i} \cdot \bar{a} = \bar{i} \cdot \bar{ae}$  in  $\mathbf{C}(S)$ . Since  $\mathbf{C}(S)$  is a group, we derive now that  $\bar{a} = \bar{ae}$ . Since  $e \in I$ , the element  $ae$  must lie in  $I \subseteq T$ . Hence  $\bar{S} = \bar{T}$  in  $\mathbf{C}(S)$ . Restricting the action of the states from  $\mathcal{C}(S)$  to  $T^*$  yields the automaton  $\mathcal{C}(T)$ . Therefore  $\mathbf{C}(T)$  is a homomorphic image of  $\mathbf{C}(S)$ , and as so is a group.

So, by the induction hypothesis,  $T$  is an inflation of a right zero semigroup by null semigroups. Suppose without loss of generality that  $a \in A_k$ . Then  $A_1, \dots, A_{k-1}, A_k \setminus \{a\}$  are the correspondent null semigroups from  $T$ . For each  $i$ , let  $e_i \in A_i$  be the right zero in  $A_i$ . Then  $S^2 = \{e_1, \dots, e_k\}$ . In particular,  $e_k \neq a$  since  $a$  is indecomposable. Take now  $a_i \in A_i$  and  $a_j \in A_j$ . Recall that  $sx = sy$  as soon as  $x$  and  $y$  are from the same kernel class.

- (1) If  $a_i \neq a$  and  $a_j \neq a$ , then  $a_i a_j = e_i e_j = e_j$ .
- (2) If  $a_i \neq a$  and  $a_j = a$ , then  $a_i a_j = a_i a = a_i e_k = e_k$ .
- (3) Let  $a_i = a$  and  $a_j \neq a$ . Let  $aa_j = e_m$  for some  $m$ . Then  $e_m = e_m^2 = e_m a a_j$ . Since  $e_m a \in T$ , it follows that  $e_m a a_j = e_j$  and so  $e_m = e_m a a_j = e_j$ . Hence  $a_i a_j = e_j$ .
- (4) If  $a_i = a_j = a$ , then  $a_i a_j = a^2 = a e_k = e_k$ .

Thus  $S$  is an inflation of a right zero semigroup  $\{e_1, \dots, e_k\}$  and the induction step is established.  $\square$

#### 4. PROOF OF THEOREM 1.3

*Proof of Theorem 1.3.* ( $\Rightarrow$ ). Suppose that  $\mathbf{C}(S)$  is finite. Take any  $\mathcal{H}$ -class  $H$  in  $S$ . With the seek of a contradiction, suppose that  $|H| > 1$ . Let  $T = \{t \in S : tH \subseteq H\}$ . Then for every  $t \in T$ , by [1, Lemma 2.21], the mapping  $\gamma_t : h \mapsto th$ ,  $h \in H$ , is a bijection of  $H$  onto itself. The set of all these bijections forms the so-called *dual Schützenberger group*  $\Gamma^*(H)$  of  $H$ . By [1, Theorem 2.22] we have  $|\Gamma^*(H)| = |H|$ . Let  $\Delta(H)$  be the dual group of  $\Gamma^*(H)$ : that is has the same underlying set as  $\Gamma^*(H)$  but in  $\Delta(H)$  we have  $\gamma_x \circ \gamma_y = \gamma_{xy}$  for all  $x, y \in T$ .

Take arbitrary  $t_1, \dots, t_k \in T$ . Then for all  $x \in H$ :

$$q(\overline{t_1} \cdots \overline{t_k}, x) = \overline{t_1 x} \cdot \overline{t_2 t_1 x} \cdots \overline{t_k \cdots t_1 x}.$$

We also have  $\tau(\overline{t_1} \cdots \overline{t_k}) = \tau(\overline{t_k \cdots t_1})$ .

Take now  $\gamma_{t_1}, \dots, \gamma_{t_k}, \gamma_x \in \mathcal{C}(\Delta(H))$ . Then

$$q(\overline{\gamma_{t_1} \cdots \gamma_{t_k}}, \gamma_x) = \overline{\gamma_{t_1} \circ \gamma_x} \cdot \overline{\gamma_{t_2} \circ \gamma_{t_1} \circ \gamma_x} \cdots \overline{\gamma_{t_k} \circ \cdots \circ \gamma_{t_1} \circ \gamma_x}$$

and  $\tau(\overline{\gamma_{t_1}} \cdots \overline{\gamma_{t_k}}) = \tau(\overline{\gamma_{t_k} \circ \cdots \circ \gamma_{t_1}}) = \tau(\overline{\gamma_{t_k \cdots t_1}})$ .

Take  $t \in T$  and consider the restriction of  $\bar{t}$  to  $H^*$ . From the very definition of  $\Gamma^*(H)$ , it now follows that the mapping  $\bar{t} \mapsto \overline{\gamma_t}$  gives rise to a well-defined homomorphism from  $\langle \bar{t} \mid_{H^*} : t \in T \rangle$  onto  $\mathbf{C}(\Delta(H))$ . It means that  $\langle \bar{T} \rangle$  has a free semigroup on  $|\Delta(H)| = |\Gamma^*(H)| = |H|$  points, as a homomorphic image, and so  $\langle \bar{T} \rangle$  is infinite. Thus  $\mathbf{C}(S)$  is infinite, a contradiction.

( $\Leftarrow$ ). We will prove by induction on  $|S|$  that if  $S$  is  $\mathcal{H}$ -trivial then  $\mathbf{C}(S)$  is finite. The base case  $|S| = 1$  is obvious. Assume that we have proved this for all  $\mathcal{H}$ -trivial semigroups of size  $\leq n$ . Take now any  $\mathcal{H}$ -trivial semigroup  $S$  with  $|S| = n + 1$ . Let  $M$  be the set of all maximal  $\mathcal{D}$ -classes from  $S$  and let  $I$  be the complement of all these  $\mathcal{D}$ -classes in  $S$ . If  $I$  is empty then  $M$  consists only of one  $\mathcal{D}$ -class and then  $S$  is simple. Since it is  $\mathcal{H}$ -trivial we have that  $S = L \times R$  is a rectangular band, where  $L$  is some left zero semigroup and  $R$  is some right zero semigroup. Combining Lemmas 2.2 and 2.3, we have that  $\mathbf{C}(S)$  is a right zero semigroup on  $|L|$  points and so is finite. So in the remainder of the proof we may assume that  $I \neq \emptyset$ . Notice that  $I$  is an ideal in  $S$ .

**Step 1:  $\langle \bar{I} \rangle$  is finite.**

It suffices to prove that there are finitely many products  $\bar{i} \cdot \bar{i}_1 \cdots \bar{i}_k \in \langle \bar{I} \rangle$  for any fixed  $i \in I$ . We have that  $\bar{i} \cdot \bar{i}_1 \cdots \bar{i}_k$  and  $\bar{i} \cdot \bar{j}_1 \cdots \bar{j}_n$  are distinct if and only if the restrictions of  $\bar{i}_1 \cdots \bar{i}_k$  and  $\bar{j}_1 \cdots \bar{j}_n$  on  $S^\infty \bar{i}$  coincide. Notice that  $S^\infty \bar{i} \subseteq I^\infty$ . Obviously  $\bar{i}_1 \cdots \bar{i}_k$  and  $\bar{j}_1 \cdots \bar{j}_n$  act on  $I^\infty$  in the same way as the correspondent products from  $\mathbf{C}(I)$  do. Now the claim of Step 1 follows from the induction hypothesis.

**Step 2:  $\bar{I}\langle \bar{S} \rangle$  is finite.**

Take a typical element  $\bar{i} \cdot \bar{a}_1 \cdots \bar{a}_k \in \bar{I}\langle \bar{S} \rangle$ . Then for all  $x \in S$ , we have

$$q(\bar{i} \cdot \bar{a}_1 \cdots \bar{a}_k, x) = \bar{i}x \cdot \bar{a}_1 x \cdots \bar{a}_k x.$$

Having that  $I$  is an ideal in  $S$ , we deduce that  $|\bar{I}\langle \bar{S} \rangle| \leq |I| \cdot |\langle \bar{I} \rangle|^{|S|}$ .

**Step 3:  $\langle \bar{S} \setminus \bar{I} \rangle$  is finite.**

We need to prove that there are only finitely many distinct products  $\bar{a}_1 \cdots \bar{a}_k$ , where all  $a_1, \dots, a_k$  lie in  $S \setminus I$ . Take such  $a_1, \dots, a_k$ . We have that

$$(4.1) \quad q_x = q(\bar{a}_1 \cdots \bar{a}_k, x) = \bar{a}_1 x \cdot \bar{a}_2 x \cdots \bar{a}_k x.$$

Obviously, to prove Step 3, it suffices to establish that there only finitely many such expressions (4.1). It follows immediately from Step 2, that there are finitely many such expressions with  $a_1 x \in I$ .

Denote by  $\mathcal{D}_M$  the restriction of  $\mathcal{D}$  to  $S \setminus I$ . Notice that if  $(a_i \cdots a_1 x, a_i \cdots a_1) \notin \mathcal{D}_M$ , for some  $i$ , then  $a_i \cdots a_1 x \in I$ . Indeed, if  $a_i \cdots a_1 x \in S \setminus I$ , then  $a_i \cdots a_1 \in S \setminus I$  and  $x \in S \setminus I$ . But since  $D_{a_i \cdots a_1 x} \leq D_{a_i \cdots a_1}$  and  $D_{a_i \cdots a_1 x} \leq D_x$ , we now have that  $a_i \cdots a_1 \mathcal{D} x \mathcal{D} a_i \cdots a_1 x$  (all three elements  $a_i \cdots a_1$ ,  $x$  and  $a_i \cdots a_1 x$  lie in maximal  $\mathcal{D}$ -classes). In particular, if  $(a_1 x, a_1) \notin \mathcal{D}_M$  then  $a_1 x \in I$  and so  $q_x \in \bar{I}\langle \bar{S} \rangle \cup \bar{I}$ .

From the above it follows that it suffices to prove that for a fixed  $x \in S$  with  $a_1 x \mathcal{D} a_1$  (which is the same as  $a_1 x \mathcal{D}_M a_1$ ), there exist only finitely many expressions  $q_x$ . Let  $m$  be the maximum number such that

$$(4.2) \quad a_i \cdots a_1 \mathcal{D}_M a_i \cdots a_1 x, \quad 1 \leq i \leq m.$$

Since  $a_{m+1} \cdots a_1 x \in I$ , by Step 2, it suffices to prove that there are finitely many products  $\bar{a}_1 \cdots \bar{a}_m$  with (4.2). Consider such one. We have  $a_i \cdots a_1 \mathcal{D}_M x$  for all  $i \leq m$ . In particular then we have  $a_m \cdots a_1, \dots, a_1$  are all from the same  $\mathcal{D}$ -class (in  $M$ ). This implies  $a_1 \mathcal{D} a_2 \mathcal{D} \cdots \mathcal{D} a_m$ . Then, by Lemma 2.5,  $a_1 \mathcal{L} a_2 a_1 \mathcal{L} \cdots \mathcal{L} a_m \cdots a_1$ . Since  $\mathcal{L}$  is a right congruence, we then have that  $a_1 x \mathcal{L} a_2 a_1 x \mathcal{L} \cdots \mathcal{L} a_m \cdots a_1 x$ .

Recall that it is enough to prove that there are only finitely many products  $\overline{a_1x} \cdot \overline{a_2a_1x} \cdots \overline{a_m} \cdots \overline{a_1x}$  with (4.2).

Thus it suffices to prove that there exist only finitely many different products  $\overline{a_1} \cdots \overline{a_n}$  such that  $a_1, \dots, a_n$  all come from the same  $\mathcal{L}$ -class inside a  $\mathcal{D}$ -class from  $M$ . Obviously it suffices to prove this for a fixed  $\mathcal{D}$ -class  $D$  in  $M$ .

So, we need to prove that the set

$$P = \{\overline{a_1} \cdots \overline{a_n} : a_1 \mathcal{L} a_2 \mathcal{L} \cdots \mathcal{L} a_n, a_1 \in D\}$$

is finite and then Step 3 is established.

Again we consider for  $x \in S$  the elements  $q_x = \overline{a_1x} \cdot \overline{a_2a_1x} \cdots \overline{a_n} \cdots \overline{a_1x}$ , for  $\overline{a_1} \cdots \overline{a_n} \in P$ . It suffices to prove that there are finitely many such  $q_x$ -s.

If  $a_1x \notin D$ , then  $q_x \in \overline{I\langle\overline{S}\rangle} \cup \overline{I}$ . So we may assume that  $x \in S$  is such that  $a_1x \in D$ . Find the maximum  $m$  such that  $a_i \cdots a_1, a_i \cdots a_1x \in D$  for all  $i \leq m$ . Since  $a_{m+1} \cdots a_1x \in I$ , as before, it suffices to prove that there are finitely many products  $\overline{a_1x} \cdot \overline{a_2a_1x} \cdots \overline{a_m} \cdots \overline{a_1x}$ .

We have  $a_2a_1, \dots, a_ma_{m-1} \in D$ . Now, for all  $j \leq m-1$ ,  $a_{j+1}\mathcal{D}a_{j+1}a_j\mathcal{D}a_j$  and so by Lemma 2.5 we have that  $a_{j+1}a_j \in R_{a_{j+1}} \cap L_{a_j}$ . By Clifford-Miller Theorem we obtain that  $L_{a_{j+1}} \cap R_{a_j}$  contains an idempotent. Since  $S$  is  $\mathcal{H}$ -trivial and  $a_1\mathcal{L} \cdots \mathcal{L} a_m$  we have that  $a_j$  is an idempotent and  $a_{j+1}a_j = a_{j+1}$ . It follows that  $\{a_1, \dots, a_{m-1}\}$  forms a left zero subsemigroup in  $S$ .

By the token as above, we have that it suffices to show that there are finitely many products  $Q \subseteq P$  of the type  $\overline{a_1} \cdots \overline{a_n}$  with  $a_1, \dots, a_n \in D$  and all of them lying in the same  $\mathcal{L}$ -class and forming a left zero semigroup.

We will prove by induction on  $k$  that the subset  $Q_k \subseteq Q$ , consisting of those products  $\overline{a_1} \cdots \overline{a_n}$  such that there are precisely  $k$  idempotents among  $a_1, \dots, a_n$ , is finite. This will then prove Step 3.

#### Base of induction.

$k = 1$ . To prove the base case, it is enough to show that  $\overline{a}$  is of finite order for all idempotents  $a \in D$ .

Let  $a$  be an arbitrary idempotent from  $D$ . We have  $q(\overline{a}^n, x) = \overline{ax}^n$ . If  $ax \notin D$ , then  $q_x \in \overline{I\langle\overline{S}\rangle} \cup \overline{I}$ . Let  $ax \in D$ . Consider  $q_{x,y} = q(q_x, y)$ . If  $ax$  is not an idempotent then  $q_{x,y} = \overline{axy} \cdot \overline{axaxy} \cdots \overline{(ax)^ny} \in \overline{S} \cdot \overline{I} \cup \overline{S} \cdot \overline{I\langle\overline{S}\rangle}$ .

Let  $E$  be the set of all idempotents  $\mathcal{R}$ -equivalent to  $a$ . Let  $X_a$  be the set of all  $x \in D$  such that  $ax$  is an idempotent in  $D$ . Notice that for  $x \in D$ ,  $ax \in D$  if and only if  $L_a \cap R_x$  is an idempotent.

Since the set  $\overline{I} \cup \overline{I\langle\overline{S}\rangle} \cup \overline{S} \cdot \overline{I\langle\overline{S}\rangle} \cup \overline{S} \cdot \overline{I}$  is finite, we have that there are only finitely many elements  $q(\overline{a}^n, x)$  with  $x \in S \setminus X_a$ . On the other hand,  $q(\overline{a}^n, x) = \overline{ax}^n$  and  $ax \in E$ , for all  $x \in X_a$ . Thus, by wreath recursions for all elements  $\overline{a}^n$ ,  $a \in E$ , we have that  $\overline{a}$  is of finite order for every idempotent  $a \in D$ . Hence the base case is established.

#### Induction step.

We will do step  $k \mapsto k+1$ . Take an arbitrary product  $\pi = \overline{a_1} \cdots \overline{a_n} \in Q_{k+1}$ . There are precisely  $(k+1)$  different  $\mathcal{R}$ -classes among  $R_{a_1}, \dots, R_{a_n}$ . Obviously, it would suffice to prove the step if  $a_1, \dots, a_n$  come from fixed  $(k+1)$   $\mathcal{R}$ -classes (and for every of these  $\mathcal{R}$ -classes there is at least one representative among  $a_1, \dots, a_n$ ). In particular, in the remainder of the proof all the products from  $Q_{k+1}$  will involve these fixed  $\mathcal{R}$ -classes.

With every such product  $\pi$  we associate the correspondent  $\mathcal{L}$ -class  $L(\pi) = L_{a_1} = \cdots = L_{a_n}$ . We have  $q_x = q(\pi, x) = \overline{a_1x} \cdots \overline{a_nx}$  for all  $x \in S$ . Notice that if  $a_1x \in D$  then  $a_1x\mathcal{L} \cdots \mathcal{L} a_nx$  and  $a_i\mathcal{R}a_ix$ . In addition, for every  $\mathcal{L}$ -class in  $D$  there exists  $x$  such that  $a_1x$  lies in this  $\mathcal{L}$ -class.

Now we split  $S$  into three disjoint sets:

- The set  $A(\pi)$  of all  $x$  such that  $a_1x \notin D$ .
- The set  $B(\pi)$  of all  $x$  such that  $a_1x \in D$  and there are at most  $k$  idempotents among  $a_1x, \dots, a_nx$ .
- The set  $C(\pi)$  of all  $x$  such that  $a_1x \in D$  and there are precisely  $(k+1)$  idempotents among  $a_1x, \dots, a_nx$ .

Notice that  $a_1x \in D$  if and only if  $L(\pi) \cap R_x$  is an idempotent. Thus each of  $A(\pi), B(\pi), C(\pi)$  depends only on  $L(\pi)$ .

If  $x \in A(\pi)$  then  $q_x \in \bar{I} \cup \bar{I}\langle \bar{S} \rangle$ .

Let  $x \in B(\pi)$ . Take  $y \in S$ . We have  $q(q_x, y) = \overline{a_1xy} \cdots \overline{a_nxa_{n-1}x \cdots a_1xy}$ . Let  $m$  be maximum such that  $a_ix \cdots a_1xy \in D$  for all  $i \leq m$ . Recall that  $a_1x\mathcal{L} \cdots \mathcal{L}a_nx$ . So, as before, we have that  $a_1x, \dots, a_{m-1}x$  are idempotents. There are at most  $k$  such idempotents and so  $a_1x, \dots, a_{m-1}x$  split in at most  $k$   $\mathcal{R}$ -classes. We have  $a_ix \cdots a_1xy = a_ixy\mathcal{R}a_ix$  and so there are at most  $k$  idempotents among  $a_1xy, \dots, a_{m-1}xy$ . Thus for all  $y \in S$ , we have  $q(q_x, y) \in \bar{I} \cup \bar{I}\langle \bar{S} \rangle \cup Q_k\bar{S}(\bar{I} \cup \bar{I}\langle \bar{S} \rangle)$ . This implies that there are only finitely many  $q_x$  for every  $\pi \in Q_{k+1}$  and  $x \in B(\pi)$ .

Let, finally,  $x \in C(\pi)$ . Since  $a_1x, \dots, a_nx$  lie in exactly  $(k+1)$   $\mathcal{R}$ -classes, we have that all of  $a_1x, \dots, a_nx$  are idempotents. In particular  $q_x = \overline{a_1x} \cdots \overline{a_nx} \in Q_{k+1}$  and  $a_1x, \dots, a_nx$  involve the same (fixed)  $\mathcal{R}$ -classes as  $a_1, \dots, a_n$ . We also mention that if we fix some  $\mathcal{L}$ -class  $L$  in  $D$  such that  $L = L(\rho)$  for some  $\rho \in Q_{k+1}$ , then the set of all  $q(\pi, x)$ , where  $\pi \in Q_{k+1}$ ,  $L = L(\pi)$  and  $x \in C(\pi)$ , exhausts the whole of  $Q_{k+1}$ .

Let  $M$  be the total (finite) number of elements in  $\{q(\pi, x) : x \in A(\pi) \cup B(\pi), \pi \in Q_{k+1}\}$ . Let also  $N = M^{|S|}|S|$  and  $p$  be the number of  $\mathcal{L}$ -classes in  $D$ .

Take now any product  $\pi = \overline{a_1} \cdots \overline{a_{N^p+1}} \in Q_{k+1}$ . We will prove that  $\pi$  equals some element from  $Q_{k+1}$  of length less than  $N^p + 1$ . That will complete the induction step and the whole proof of Step 3.

Let  $L_1, \dots, L_q$  be all  $\mathcal{L}$ -classes, which in intersection with the fixed  $\mathcal{R}$ -classes give  $(k+1)$  idempotents. Assume without loss of generality that  $L(\pi) = L_1$ . Note that  $\pi = q(\pi, x)$  for some  $x \in C(\pi)$ .

By Pigeonhole Principle, we have that there exist  $1 \leq i_1 < \cdots < i_{N^{p-1}+1} \leq N^p + 1$  such that  $\tau(\overline{a_1} \cdots \overline{a_{i_j}}) = \tau(\overline{a_1} \cdots \overline{a_{i_k}})$  and  $q(\overline{a_1} \cdots \overline{a_{i_j}}, x) = q(\overline{a_1} \cdots \overline{a_{i_k}}, x)$  for all  $x \in A(\pi) \cup B(\pi)$  and  $j < k$ . Notice now that  $L(\pi) = L(\overline{a_{i_1}} \cdots \overline{a_{i_k}})$ . There exists  $y \in C(\pi)$  such that  $L(q(\pi, y)) = L_2$ . Analogously, by Pigeonhole Principle, we have that there is a subsequence  $i_1 \leq j_1 < \cdots < j_{N^{p-2}+1} \leq i_{N^{p-1}+1}$  such that  $\tau(\overline{a_{i_1}y} \cdots \overline{a_{j_u}y}) = \tau(\overline{a_{i_1}y} \cdots \overline{a_{j_v}y})$  and  $q(\overline{a_{i_1}y} \cdots \overline{a_{j_u}y}, x) = q(\overline{a_{i_1}y} \cdots \overline{a_{j_v}y}, x)$  for all  $x \in A(q(\pi, y)) \cup B(q(\pi, y))$  and  $u < v$ . Proceeding in this way in total at most  $q$  times we arrive at two indices  $u < v$ , such that

$$\tau(\overline{a_1z} \cdots \overline{a_uz}) = \tau(\overline{a_1z} \cdots \overline{a_vz})$$

and

$$q(\overline{a_1z} \cdots \overline{a_uz}, x) = q(\overline{a_1z} \cdots \overline{a_vz}, x)$$

for all  $z \in C(\pi)$ ,  $x \in A(q(\pi, z)) \cup B(q(\pi, z))$ .

Finally, we remark that if  $x \in C(\pi)$  and  $y \in C(q(\pi, x))$ , then  $xy \in C(\pi)$ . Thus, from wreath recursions for elements  $\overline{a_1z} \cdots \overline{a_uz}$  and  $\overline{a_1z} \cdots \overline{a_vz}$  for all  $z \in C(\pi)$ , it now follows that  $\overline{a_1} \cdots \overline{a_u} = \overline{a_1} \cdots \overline{a_v}$  and so

$$\pi = \overline{a_1} \cdots \overline{a_u} \cdot \overline{a_{v+1}} \cdots \overline{a_{N^p+1}}$$

is of length strictly less than  $N^p + 1$ .

Therefore the induction step is proved and so Step 3 is established.

#### Step 4: $\langle \bar{S} \rangle$ is finite.

We have

$$\langle \bar{S} \rangle = \bar{I}\langle \bar{S} \rangle^1 \cup \langle \bar{S} \setminus \bar{I} \rangle \cup \langle \bar{S} \setminus \bar{I} \rangle \bar{I}\langle \bar{S} \rangle^1$$

is finite by Steps 2 and 3.  $\square$

## 5. PROOF OF PROPOSITION 1.4

*Proof of Proposition 1.4.* ( $\Rightarrow$ ). Suppose that  $\mathbf{C}(S)$  is free. Let  $K$  be the minimal ideal of  $S$ . Then  $K$  is a Rees matrix semigroup  $\mathcal{M}[G; I, J; P]$  for some  $J \times I$ -matrix  $P$  and group  $G$  with identity  $e$ . By [4, Theorem 3.4.2] we even may assume that  $1 \in I$ ,  $1 \in J$  and  $p_{j1} = p_{1i} = e$  for all  $i \in I$ ,  $j \in J$ . Then the element  $k = (1, e, 1) \in K$  is clearly an idempotent. Then  $sk = sk^2$  and  $sk \in I$  for all  $s \in S$ . Therefore, by wreath recursions,  $\overline{k} \cdot \overline{s} = \overline{k} \cdot \overline{sk}$ . Since a free semigroup is left cancellative, we obtain  $\overline{s} = \overline{sk}$  and so, by Lemma 2.1,  $\lambda_s = \lambda_{sk}$ . Hence  $\overline{S}$  coincides with  $\overline{L_k}$ , where  $L_k$  is the  $\mathcal{L}$ -class containing  $k$ . Let  $j \in J$  and  $i \in I$ . The condition  $\lambda_{(1,e,j)} = \lambda_{(1,e,j)(1,e,1)}$  implies that (since  $p_{j1} = p_{1i} = e$ )

$$(1, p_{ji}, 1) = (1, e, j)(i, e, 1) = (1, e, j)(1, e, 1)(i, e, 1) = (1, e, 1),$$

and so  $p_{ji} = e$ . Then for all  $i, h \in I$  we have  $\overline{(i, e, 1)} \cdot \overline{(h, e, 1)} = \overline{(j, e, 1)} \cdot \overline{(h, e, 1)}$ . It follows that  $\overline{(i, e, 1)} = \overline{(h, e, 1)}$  and hence  $\lambda_{(i,e,1)} = \lambda_{(h,e,1)}$ . Thus  $i = h$  and so  $K$  contains only one  $\mathcal{R}$ -class.

Finally, since  $\lambda_s = \lambda_{sk}$  for all  $s \in S$ , we have that  $S^2 \subseteq K$ . Hence the only non-singleton  $\mathcal{H}$ -classes in  $S$  must be those lying in  $K$ . If  $K$  contains singleton  $\mathcal{H}$ -classes, then  $S$  is  $\mathcal{H}$ -trivial and so, by Theorem 1.3,  $\mathbf{C}(S)$  is finite, a contradiction. Thus all  $\mathcal{H}$ -classes in  $K$  are non-singleton.

( $\Leftarrow$ ). Since  $K$  contains only one  $\mathcal{R}$ -class, we have that  $K = G \times R$  where  $G$  is a group with identity  $e$  and  $R$  is a right zero semigroup. Let  $k = (h, s) \in K$  be as in the hypothesis. Then for every  $(g, r) \in K$ , we have  $\lambda_{(g,r)} = \lambda_{(g,r)(h,s)} = \lambda_{(gh,s)}$ . Then  $(g, t) = (g, r)(e, t) = (gh, s)(e, t) = (gh, t)$  and therefore  $h = e$  and  $k = (e, s)$ . Hence, by Lemma 2.1,  $\overline{S} = \overline{H_{(e,s)}}$ . As in the proof of Theorem 1.3, we have that  $\mathbf{C}(S)$  can be homomorphically mapped onto  $\mathbf{C}(H_{(e,s)})$ . But by Theorem 1.1,  $\mathbf{C}(H_{(e,s)})$  is free of rank  $|H_{(e,s)}|$ , so  $\mathbf{C}(S)$  is free of rank  $|H_{(e,s)}|$ .  $\square$

## 6. PROOF OF PROPOSITIONS 1.5 AND 1.6

*Proof of Proposition 1.5.* ( $\Rightarrow$ ). Suppose that  $\mathbf{C}(S)$  is a right zero semigroup. Let  $a, b \in S$ . Then  $\overline{b} \cdot \overline{a} = \overline{a}$ . In particular,  $\lambda_{ab} = \lambda_a$ . This implies that  $abc = ac$  for all  $a, b, c \in S$ .

( $\Leftarrow$ ). Suppose that  $abc = ac$  for all  $a, b, c \in S$ . Then  $\lambda_{ab} = \lambda_a$  for all  $a, b \in S$ . Now,  $\overline{b} \cdot \overline{a} = \overline{a}$  if and only if  $\lambda_{ab} = \lambda_a$  and  $\overline{bx} \cdot \overline{ax} = \overline{ax}$  for all  $x \in S$ . By hypothesis, the latter is equivalent to  $\overline{bx} \cdot \overline{ax} = \overline{ax}$ . By recursive arguments we now obtain that  $\overline{b} \cdot \overline{a} = \overline{a}$  for all  $a, b \in S$ . Thus  $\mathbf{C}(S)$  is a right zero semigroup.  $\square$

*Proof of Proposition 1.6.* ( $\Rightarrow$ ). Suppose that  $\mathbf{C}(S)$  is a left zero semigroup. Since this left zero semigroup is finitely generated, it is finite. So, by Theorem 1.3,  $S$  is  $\mathcal{H}$ -trivial. Let  $I$  be the minimal ideal in  $S$ . Then  $\langle \overline{I} \rangle \subseteq \mathbf{C}(S)$  can be homomorphically mapped onto  $\mathbf{C}(I)$ . Since  $I$  is simple and finite, it is a Rees matrix semigroup. Since  $S$  is  $\mathcal{H}$ -trivial,  $I = X \times Y$  for some left zero semigroup  $X$  and a right zero semigroup  $Y$ . By Lemmas 2.2 and 2.3,  $\mathbf{C}(I)$  is a right zero semigroup on  $|X|$  points. A homomorphic image of the left zero semigroup  $\langle \overline{I} \rangle$  must be a left zero semigroup. Hence  $|X| = 1$  and so  $I$  is a right zero semigroup.

Let  $s \in S$  and  $i \in I$ . Then, since  $\mathbf{C}(S)$  is a left zero semigroup,  $\overline{s} \cdot \overline{i} = \overline{s}$ ; consequently  $\lambda_s = \lambda_{is}$ . By Lemma 2.1,  $\overline{s} = \overline{is} \in \overline{I}$ . In particular  $S\lambda_s \subseteq I$ . Since this holds for each  $s \in S$ , it follows that  $S^2 \subseteq I$ . Thus  $I = S^2$ .

( $\Leftarrow$ ). Suppose that the minimal ideal  $I$  of  $S$  coincides with  $S^2$  and that  $I$  is a right zero semigroup.



Take an arbitrary  $s \in S$  and fix  $i \in I$ . Then for every  $x \in S$  we have that  $sx \in I$  and so  $isx = sx$ . This implies that  $\lambda_s = \lambda_{is}$ . By Lemma 2.1, we have that  $\overline{s} = \overline{is}$ . Therefore  $\overline{S} = \overline{I}$  and in particular  $\mathbf{C}(S) = \langle \overline{I} \rangle$ . So it suffices to prove that  $\overline{i} \cdot \overline{j} = \overline{i}$  for all  $i, j \in I$ . Note that if  $\alpha \in S^\infty$ , then  $\alpha \cdot \overline{i} \in I^\infty$ . Since  $I$  is a right zero semigroup,  $\overline{j}$  acts identically on  $I^\infty$ . Hence  $\alpha \cdot (\overline{i} \cdot \overline{j}) = \alpha \cdot \overline{i}$  for all  $\alpha \in S^\infty$  and so  $\overline{i} \cdot \overline{j} = \overline{i}$ , as required.  $\square$

## 7. FURTHER DISCUSSION

In Theorem 1.2 we proved that no Cayley automaton semigroup can be a non-trivial group. In addition, it is proved in [6] that if  $S$  is a finite  $\mathcal{H}$ -trivial monoid, then  $\mathbf{C}(S)$  is a (finite)  $\mathcal{H}$ -trivial semigroup. In fact, the author believes that every Cayley automaton semigroup is  $\mathcal{H}$ -trivial and poses an

**Open Problem 7.1.** Are all Cayley automaton semigroups  $\mathcal{H}$ -trivial?

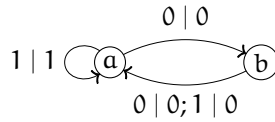
The following proposition is an important consequence of Theorem 1.3.

**Proposition 7.2.** *Any infinite Cayley automaton semigroup contains a free semigroup of rank 2.*

*Proof.* Suppose  $\mathbf{C}(S)$  is infinite. Then  $S$  is not  $\mathcal{H}$ -trivial. So  $S$  contains an  $\mathcal{H}$ -class  $H$  with  $|H| > 1$ . Then as in the proof of the necessity of Theorem 1.3, there exists a subsemigroup  $T \leq S$  such that  $\langle \overline{T} \rangle$  has a free semigroup of rank  $|H|$  as a homomorphic image.  $\square$

**Corollary 7.3.** *The free product of two trivial semigroups  $\text{Sg}\langle e, f \mid e^2 = e, f^2 = f \rangle$  and free commutative semigroups of rank  $> 1$  are all automaton semigroups, but neither of them is a Cayley automaton semigroup.*

*Proof.* That free commutative semigroups of rank  $> 1$  are automaton semigroups can be found in [5]; and it is routine to check that the automaton semigroup generated by the automaton



is isomorphic to  $\text{Sg}\langle e, f \mid e^2 = e, f^2 = f \rangle$ . That neither of these semigroups is a Cayley automaton semigroup follows immediately from Proposition 7.2.  $\square$

**Remark 7.4.** The characterization of those finite semigroups  $S$  such that  $\mathbf{C}(S)$  is a right zero semigroup, is ‘close’ to the characterization of rectangular bands: the latter are precisely those semigroups  $S$  such that all the elements from  $S$  are idempotents and  $abc = ac$  for all  $a, b, c \in S$ , [4, Theorem 1.1.3].

In the following example we show that it is possible for Cayley automaton semigroup to be a *non-trivial* left zero semigroup:

**Example 7.5.** Define a finite semigroup  $S$  on four points  $i, j, k, f$  with the following multiplication table:

	$i$	$j$	$k$	$f$
$i$	$i$	$j$	$k$	$i$
$j$	$i$	$j$	$k$	$i$
$k$	$i$	$j$	$k$	$j$
$f$	$i$	$j$	$k$	$i$

Then  $\mathbf{C}(S)$  is a left zero semigroup on 2 points.

*Proof.* One checks that the multiplication table indeed gives a semigroup. By Lemma 2.1,  $\bar{i} = \bar{j} = \bar{f}$ . Hence, by Proposition 1.6,  $\mathbf{C}(S)$  is a left zero semigroup generated by  $\bar{j}$  and  $\bar{k}$ . It remains to notice that  $\bar{j} \neq \bar{k}$ . It follows from Lemma 2.1 and  $f\lambda_j = jf = i \neq j = kf = f\lambda_k$ .  $\square$

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